A Mechanical Approach to Solve Two-Dimensional Static Electrical and Magnetic Fields: Applications to Contact Between Conductors Under Electrical Load

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A general approach has been developed to obtain analytical solutions to the boundary-value problems for a two-dimensional conductor under static electric and magnetic fields. This approach is based on a "congruity principle" between a solution of Maxwell's equation and the corresponding linear elastic plane stress solution with constant mean stress or plane strain solution with constant mean in-plane stress. It also leads to a new avenue to construct analytical solutions of antiplane strain boundary-value problems using plane stress/strain solutions, or vice versa, in linear elastic theory. This approach has been applied for such engineering problems as contact between two conducting solid bodies under electrical load. [DOI: 10.1115/1.4000037]

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1 Introduction

Although the electrostatic mean-field theory and linear elasticity belong to different branches of engineering science [1–22], the similarities between these two classes of physical phenomena and consistency in mathematical governing equations have been long noticed by researchers [8,16,23,24]. For example, a two-dimensional static electrical field in a conductor or dielectric medium is mathematically identical to the corresponding stress field of an antiplane strain elastic problem. Reviews about these similarities can be found, e.g., in Section 2.8 of Ref. [8]. However, from the perspectives of theoretical analysis of mean-field electrostatic applications, challenges remain at least in the following three respects: (i) only limited analytical solutions have been found [4], which is also true for antiplane strain problems in elastic theory [1,5,6,8–11], (ii) a magnetic field is always accompanied with an electrical field but very few reported analytical solutions can be found; it is also not clear if there is a corresponding solution to electrostatic magnetic field in linear elasticity, and (iii) no reported study has been found about the relationship between the two-dimensional (plane strain and plane stress) elastic solutions and their electrostatic counterpart.

Recently, the boundary value problems coupling mechanical and electrical-magnetic loads have gained increasing attention [15,18–21,25–33], because nanotechnology brings up rapid developments of micro- and nanoscale machines and devices [25–27,33]. Under such small scales many mechanical problems are naturally coupled with electrical and magnetic loads. On the other hand, magnetic force-based transportation systems become more common in engineering applications [28–30,34]. In both cases, the corresponding boundary value problems are usually associated with contact and friction. This class of problems can be simplified as the system illustrated in Fig. 1(*a*), where a conductor under mechanical pressure and electrical load is resting upon a semi-infinite elastic conducting substrate. The corresponding pure elastic contact solutions for various contact zone profiles have been thoroughly investigated [7,12,13,35–37]. By contrast, this issue remains open for closed form solutions under electrical loads.

Focusing on these challenges, a study has been performed in this paper to explore the intrinsic linkage between an elastic plane stress/strain solution and the corresponding boundary-value problem in electrostatic theory, which leads to a "congruity principle" between these two classes of problems and an approach to obtain analytic solutions of two-dimensional Maxwell's equations using the Airy stress function obtained by, e.g., Muskhelishvili's method [6]. It also leads to a new avenue to solve antiplane strain elastic problems analytically, based on plane stress/strain elastic solutions, or vice versa. Though numerical computation becomes the primary means, a theoretical solution able to provide an accurate overview of an engineering problem remains as a challenging and attractive object for many purposes [15–23,38,39,33–37].

In this paper standard notation is used throughout the text, except when otherwise specified. The boldface symbols denote tensors; the order of a tensor is indicated by the context. Plain symbols denote scalars or components of a tensor when a subscript is attached. Repeated indices are summed. For second-order tensors $\boldsymbol{a}$ and $\boldsymbol{b}$: $\boldsymbol{a}=\lfloor a_{ij}\rfloor$, $\boldsymbol{b}=\lfloor b_{ij}\rfloor$: $\boldsymbol{a}\cdot\boldsymbol{b}=\lfloor a_{ik}b_{kj}\rfloor$, $\boldsymbol{a}:\boldsymbol{b}=\lfloor a_{ij}b_{ij}\rfloor$, $\boldsymbol{ab}=\lfloor a_{ij}b_{kl}\rfloor$.

Regarding the coordinate system, symbol z_3 represents the coordinate perpendicular to the $\{x,y\}$ plane in a three-dimensional Cartesian system $\{x,y,z_3\}$; subscript "z" or "3" indicates a variable in the z_3 direction, whereas variable "z" represents a point in the complex plane $z=x+iy$ with $i=\sqrt{-1}$ and $\bar{z}=x-iy$ [6,8]. Notations $\mathrm{Re}\{f(z)\}$ and $\mathrm{Im}\{f(z)\}$ denote the real and imaginary parts of complex function $f(z)$, respectively, hence, when $f_1(z)$ and $f_2(z)$ are real functions:

$$f(z)=f_1(z)+if_2(z),\quad \mathrm{Re}\{f(z)\}=f_1(z),\quad \mathrm{Im}\{f(z)\}=f_2(z)$$

When both $f_1(z)$ and $f_2(z)$ can be either real or complex functions and $f(z)=f_1(z)+if_2(z)$:

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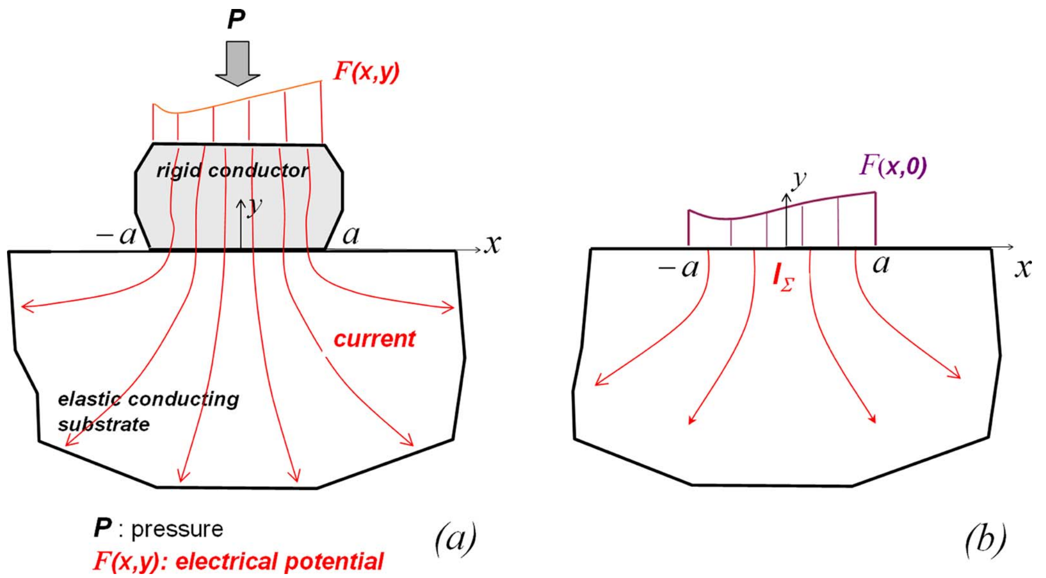

Fig. 1 The model analyzed: (*a*) a rigid conductor contacts a semi-infinite elastic conducting substrate under mechanical pressure and electrical load and the corresponding solutions of Maxwell's equations can be classified as the boundary-value problems described by (*b*)

$$\bar{f}(z)=f_1(z)-if_2(z), \quad f(\bar{z})=f_1(\bar{z})+if_2(\bar{z}) \quad \text{and}$$

$$\bar{f}(\bar{z})=\overline{f(z)}=f_1(\bar{z})-if_2(\bar{z})$$

2 Models and Governing Equations

2.1 Maxwell's Equations. Electrostatic problems are governed by Maxwell's equations. Let symbols $\boldsymbol{E}$, $\boldsymbol{J}$, $\boldsymbol{H}$, and $\boldsymbol{B}$ denote in turn macroscopic electric field, current density, macroscopic magnetic field, and magnetic induction field. For problems without source charges the macroscopic Maxwell equations in SI unit read [2,4]

$$\begin{aligned} \nabla\cdot\boldsymbol{B}&=0 \quad \text{(I)}\\ \nabla\times\boldsymbol{E}&=0 \quad \text{(II)}\\ \nabla\cdot\boldsymbol{E}&=0 \quad \text{(III)}\\ \nabla\times\boldsymbol{B}&=\mu_H\boldsymbol{J} \quad \text{(IV)}\end{aligned} \tag{1}$$

with Ohm's Law at each material point:

$$\boldsymbol{J}=\sigma^C\boldsymbol{E} \tag{2a}$$

and

$$\boldsymbol{H}=\mu_H\boldsymbol{B} \tag{2b}$$

In Eqs. (1), (2*a*), and (2*b*) magnetic permeability μ_H and electric conductivity σ^C are assumed to be constants. In this analysis only the magnetic field induced by an electric field is taken into account; therefore, the coupling effects of the magnetic field on the electric field, e.g., the Hall effect, are assumed to be relatively weak for a system as in Figs. 1(*a*) and 1(*b*) and will be omitted in the following analysis.

Under two-dimensional conditions of $E_z=B_x=B_y=0$ but $B_z\neq 0$, Maxwell equation III in Eq. (1) can be satisfied by an "electrical potential" F, a harmonic function defined as

$$E_x=-\frac{\partial F}{\partial x}, \quad E_y=-\frac{\partial F}{\partial y} \quad \text{and} \quad \nabla^2F=0 \tag{3}$$

by which the Maxwell equation II in Eq. (1) is also satisfied.

2.2 Equilibrium Condition. For a solid body obeying linear elasticity, its stress tensor $\boldsymbol{\sigma}$, strain tensor $\boldsymbol{\varepsilon}$, and displacement field $\boldsymbol{u}$, are correlated through the linear Hooke's law under small strain as follows:

$$\boldsymbol{\sigma}=\boldsymbol{C}^e:\boldsymbol{\varepsilon} \quad \boldsymbol{\varepsilon}=\frac{1}{2}[\nabla\mathbf{u}+\nabla^T\mathbf{u}] \tag{4}$$

where $\boldsymbol{C}^e$ is the elastic stiffness tensor and ∇ is the gradient operator. The static state equilibrium condition [1,6] reads

$$\nabla\boldsymbol{\sigma}=0 \tag{5}$$

when no body force is present.

Under the antiplane strain condition of $u_x=u_y=0$ and $\sigma_{xx}=\sigma_{yy}=\sigma_{xy}=\sigma_{zz}=0$, Eqs. (4) and (5), respectively, become

$$\sigma_{xz}=G\frac{\partial u_z}{\partial x}, \quad \sigma_{yz}=G\frac{\partial u_z}{\partial y} \tag{4'}$$

and

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}=0 \tag{5'}$$

where G is the shear modulus. Obviously, a solution u_z satisfying Eqs. (4′) and (5′) has its counterpart of electrical potential $F=-u_zk$ satisfying Eq. (3), where k is an arbitrary constant.

Under plane strain ($\varepsilon_{xz}=\varepsilon_{yz}=\varepsilon_{zz}=0$) or plane stress conditions ($\sigma_{xz}=\sigma_{yz}=\sigma_{zz}=0$), in Cartesian coordinates the equilibrium condition (5) and Cauchy geometric (strain-displacement) relation, respectively, become

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0, \quad \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0 \tag{6}$$

and

$$\varepsilon_{xx}=\frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy}=\frac{\partial u_y}{\partial y}, \quad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) \tag{7}$$

The corresponding elastic stiffness tensor is

$$\boldsymbol{C}^e = \frac{E'}{1-v'^2}\begin{bmatrix} 1 & v' & \\ v' & 1 & \\ & & \frac{1-v}{2} \end{bmatrix} \quad (4'')$$

for plane stress: $E'=E$, $v'=v$; plane strain: $E'=E/(1-v^2)$ and $v'=v/(1-v)$; where E is the Young's modulus and v is the Poisson's ratio.

For a linear elastic solid under small deformation:

$$(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = \frac{E}{1-2v}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) \quad (8)$$

2.3 Airy Stress Function. Equation (6) can be satisfied if stresses are expressed as the derivatives of Airy stress function U [1,6]:

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial y\, \partial x} \quad (9)$$

where U is a biharmonic function:

$$\nabla^2\nabla^2 U = 0 \quad (10)$$

According to Love [1] and Muskhelishvili [6], the corresponding displacement solution yields

$$u_x + iu_y = -\frac{1}{2G}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) + \frac{\lambda+2G}{G(\lambda+2G)}\varphi(z) \quad (11)$$

where

$$\lambda = \frac{Ev}{(1+v)(1-2v)}, \quad G = \frac{E}{2(1+v)}$$

λ and G are the Lame's constants, and $\varphi(z)$ is a "holomorphic" function in a given domain defined below as

$$\varphi(z) = \frac{1}{4}\int f(z)dz, \quad f(z) = f_1(x,y) + if_2(x,y) \quad \text{and}$$
$$f_1(x,y) = \nabla^2 U \quad (12)$$

$f(z)$ is also a holomorphic function.

Here a holomorphicfunction, $f(z)$, means that the following conditions are satisfied [6].

(i) A one-to-one mapping exists between the domain $z=x+iy, z\in\Omega$ and the mapping domain $f(z)=f_1(x,y)+if_2(x,y)$, $f(z)\in\Omega_f$.

(ii) $f(z)$ is continuous and has infinite order of derivatives that are also holomorphic, i. e., $f(z)$ can be expanded into a Laurent series:

$$f(z) = \cdots \frac{c_{-n}}{z^n} + \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots \quad (13)$$

(iii) The real and imaginary parts of $f(z)$ satisfy the Cauchy–Riemann conditions:

$$\frac{\partial f_1(x,y)}{\partial x} = \frac{\partial f_2(x,y)}{\partial y}, \quad \frac{\partial f_1(x,y)}{\partial y} = -\frac{\partial f_2(x,y)}{\partial x} \quad (14)$$

which implies

$$\nabla^2 f_1(x,y) = 0, \quad \nabla^2 f_2(x,y) = 0 \quad (15)$$

Conditions (i) and (ii) are necessary and sufficient to each other, i.e., when one of them is satisfied, so is the other.

2.4 Muskhelishvili's Complex Function Solution. Muskhelishvili (Eqs. 29–34 of Ref. [6]) proved that the biharmonic condition (10) can be satisfied in a domain, Ω, when stresses and displacements are formulated through the complex functions, $\varphi(z)$, $\psi(z)$ and $\Phi(z)=\varphi'(z)$, $\Psi(z)=\psi'(z)$:

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \overline{\Phi(z)}] \quad (16)$$

$$-\sigma_{xx} + \sigma_{yy} + 2i\sigma_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \quad (17)$$

$$u_x + iu_y = \frac{1}{2G}[\kappa\varphi(z) - z\overline{\varphi(z)} - \overline{\psi(z)}] \quad (18)$$

and

$$U = \tfrac{1}{2}[z\overline{\varphi(z)} + \bar{z}\varphi(z) + \psi(z) + \overline{\psi(z)}] \quad (19)$$

where

$$\kappa = \frac{\lambda+3G}{\lambda+2G}$$

and $(\ )' = d/dz(\ )$; $\varphi(z)$ and $\psi(z)$ are holomorphic functions within the domain, Ω, and $\varphi(z)$ is related to the Airy stress function, U, in Eqs. (19) and (12).

3 The Method of Approach

As mentioned previously, the structure of the antiplane strain solution of Eqs. (4′) and (5′) is the same as that of Eq. (3), provided the out-plane elastic displacement, u_z, is replaced by electrical potential F. By contrast, no study has reported the connection between the electromagnetic field governed by Eqs. (1), (2*a*), (2*b*), and (3) and the plane strain stress field governed by Eqs. (6)–(8), which is the objective of this research.

According to Eq. (1) the satisfaction of Eq. (3) requires the electrical potential F to be harmonic; whereas the Airy stress function is "biharmonic" and the displacement solutions u_x and u_y in Eq. (18) are generally not harmonic, except under the additional constraint that is defined by a "constant mean in-plane stress rule."

3.1 Constant Mean In-Plane Stress Rule. When a plane stress (or plane strain) linear elastic solution satisfies the following condition:

$$\sigma_{xx} + \sigma_{yy} = \text{const} \quad (20)$$

the corresponding displacement solutions $\{u_x, u_y\}$ are harmonic, i.e., either of them can be used as a two-dimensional electrostatic potential function in Eq. (3) with corresponding boundary condition.

This rule can be verified since, according to Eq. (9), the constraint (20) implies

$$\nabla^2 U = \text{const} \quad (21a)$$

Performing operation ∇^2 to the displacement field (11) and substituting (21*a*) into the resulting equation leads to

$$\nabla^2(u_x + iu_y) = 0 \quad \text{or} \quad \nabla^2 u_x = 0 \quad \nabla^2 u_y = 0 \quad (21b)$$

i.e., both u_x and u_y are harmonic.

It should be noted that constraint (20) is a necessary condition for the incompressibility under plane stress state, but it may not be true for plane strain due to the nonzero out-plane stress σ_{zz}. Nevertheless, any linear elastic plane stress solution is structurally identical to a corresponding plane strain solution but with different elastic stiffness coefficients.

The "constant mean in-plane stress rule" implies that each linear elastic plane stress solution with constant mean stress has its counterpart in the pool of electrostatic solutions satisfying Eq. (3), by which the electrical potential can be expressed either as $F=-u_x k$ or $F=-u_y k$, where k is an arbitrary constant. It also leads to a new way to construct an analytical solution of an electrostatic boundary-value problem by applying a linear elastic solution with similar boundary condition without satisfying Eq. (20) and can be stated by the following "conjugate solution construction formula."

3.2 Conjugate Solution Formula. A plane stress (or plane strain) displacement solution, which is constructed by a holomorphic stress function $\varphi(z)$ $(=\varphi_1(x,y)+i\varphi_2(x,y))$ in the form of Eqs. (16)–(18) but may not satisfy Eq. (20), defines the following two harmonic functions which are the solutions of two-dimensional static Maxwell equation III of Eq. (1):

$$F_1 + iF_2 = k\left\{\frac{1}{2}\left(\frac{\partial U_E}{\partial x} \pm i\frac{\partial U_E}{\partial y}\right) - \varphi_E(z)\right\} \quad (22)$$

where F_1 and F_2 are harmonic functions, which can be used as electrical potentials corresponding to different boundary conditions, k is an arbitrary constant, $\varphi_E(z)$ is the conjugate function of $\varphi(z)$, so $\varphi_E(z)$ and its derivative $\Phi_E(z)$ are defined below.

$$\Phi_E(z) = \varphi_E'(z) = \overline{\varphi'(z)} \quad \text{and} \quad \varphi_E(z) = \overline{\varphi(z)} \quad (23)$$

The electrical Airy stress function $U_E(z)$ in Eq. (22) is defined by

$$U_E = \tfrac{1}{2}[z\overline{\varphi_E(z)} + \bar{z}\varphi_E(z) + \psi_E(z) + \overline{\psi_E(z)}] \quad (24)$$

where $\psi_E(z)=\psi_{E1}(x,y)+i\psi_{E2}(x,y)$ and $\psi_{E1}(x,y)$, $\psi_{E2}(x,y)$ are harmonic.

The plane stress/strain solution defined by Eqs. (16)–(18) belongs to those constructed by the holomorphic function $\varphi(z)$ using Muskhelishvili's method [6]. However, since $\varphi_E(z)$ in Eqs. (22)–(24) may not be holomorphic, it has to be verified that $\varphi_E(z)$ and $U_E(z)$ also define a plane stress solution $\sigma_{ij}(U_E)$ that satisfies Eqs. (20) and (21*b*), although the solution defined by $\varphi(z)$ may not meet this constraint. Alternatively, the relationship (24) can be rewritten as

$$U_E = x\varphi_1(x,y) - y\varphi_2(x,y) + \psi_{E1}(x,y) \quad (25)$$

since ψ_{E1}, ψ_{E2}, φ_1, and φ_2 are harmonic while φ_1 and φ_2 obey Cauchy–Riemann condition (14), then,

$$\nabla^2 U_E = \frac{\partial\varphi_1}{\partial x} - \frac{\partial\varphi_2}{\partial y} + \nabla^2\varphi_1 - \nabla^2\varphi_2 + \nabla^2\psi_{E1} = 0 \quad (26)$$

By substituting Eq. (25) into Eq. (9), Eq. (20) is satisfied.

$$\sigma_{xx}(U_E) + \sigma_{yy}(U_E) = \nabla^2 U_E = 0$$

Since $U_E(z)$ is also biharmonic, applying Eq. (11), $\varphi_E(z)$ and $U_E(z)$ determine the following displacement solution of Eq. (5):

$$u_x + iu_y = -\frac{1}{2G}\left(\frac{\partial U_E}{\partial x} + i\frac{\partial U_E}{\partial y}\right) + \frac{\lambda + 2G}{G(\lambda + 2G)}(\varphi_1 - i\varphi_2) \quad (27)$$

Both the real and imaginary parts on its right hand side are harmonic.

No attention thus far has been given to the magnetic field. It has been proven in [34] that, under the two-dimensional conditions of $E_z=H_x=H_y=0$, a plane electric potential F and the corresponding current-induced magnetic field H_z, which is perpendicular to the two-dimensional plane, are conjugate to each other, i.e., they form a holomorphic function in a given domain:

$$f(z) = -F + i\frac{H_z}{\sigma^C} \quad (28)$$

where σ^C is the conductivity. The derivation of this relation is briefly given in the Appendix. On the other hand, a plane stress/strain elastic solution does not assure that u_x and u_y are conjugate. The following principle establishes the linkage between linear elastic plane solution and magnetic field.

3.3 Congruity Principle. A linear elastic plane stress (or plane strain) solution in the form of Eqs. (16)–(18) is congruent to the following electromagnetic field solution of two-dimensional static Maxwell equations (1) with the difference in constant factor k:

$$-F + i\frac{H_z}{\sigma^C} = k\left\{\frac{1}{2}\left(\frac{\partial U_E}{\partial x} - i\frac{\partial U_E}{\partial y}\right) + \bar{\varphi}_E(\bar{z})\right\} \quad (29)$$

where the stress functions $\varphi_E(z)$ and $U_E(z)$ are defined by Eqs. (23) and (24), respectively.

According to Ref. [6], any linear elastic small strain plane stress (or plane strain) solution can be expressed in the form of Eqs. (16)–(18), constructed by holomorphic function $\varphi(z)$; and $\bar{\varphi}_E(\bar{z})$ is holomorphic since $\bar{\varphi}_E(\bar{z})=\varphi(z)$ according to Eq. (23). Hence, the congruity principle is true if the first two terms on the right hand side of Eq. (29) also form a holomorphic function. This can be verified through the satisfaction of Cauchy–Riemann conditions (14), which is obvious since the first relation of Eq. (14) requires $\nabla^2 U_E(z)=0$ and the second requires $(\partial/\partial y)(\partial U_E/\partial x)=-(\partial/\partial x)\times(-\partial U_E/\partial y)$.

By comparison, between the right hand side of Eq. (29) and that of Eq. (11), one can find

$$-F + i\frac{H_z}{\sigma^C} = k(u_{Ex} - iu_{Ey}) \quad (29')$$

where u_{Ex} and u_{Ey} are the two components of a displacement field solution constructed by $\varphi_E(z)$ and $U_E(z)$.

3.4 Discussions. For anisotropic conductors, i.e., either the magnetic permeability μ_H in Eq. (1) or the electric conductivity σ^C in Eq. (2) or both of them become tensors, according to Eq. (3) the conclusions obtained previously are still applicable except the congruity principle. The relationship (A3) in the Appendix is thus no longer equivalent to the Cauchy–Riemann conditions between F and H_z, if σ^C is not a constant scalar. Instead, it defines a group of partial difference equations to determine magnetic field H_z:

$$\frac{\partial H_z}{\partial y} = -\frac{\partial F}{\partial x}\sigma^C_{xx} - \frac{\partial F}{\partial y}\sigma^C_{xy}, \quad \frac{\partial H_z}{\partial x} = \frac{\partial F}{\partial x}\sigma^C_{xy} + \frac{\partial F}{\partial y}\sigma^C_{yy} \quad (30a)$$

where

$$\sigma^C = \begin{bmatrix} \sigma^C_{xx} & \sigma^C_{xy} \\ \sigma^C_{xy} & \sigma^C_{yy} \end{bmatrix} \quad (30b)$$

and μ_H is assumed to be a constant scalar.

For anisotropic dielectrical materials, Maxwell's equation III of Eq. (1) becomes

$$\nabla \cdot \mathbf{D} = 0 \quad (31a)$$

where

$$\mathbf{D} = \boldsymbol{\varepsilon}:\mathbf{E} \quad (31b)$$

where $\mathbf{D}$ is the electrical induction and $\boldsymbol{\varepsilon}$, the dielectric coefficient, is a second-order tensor after ignoring higher-order terms. Under these conditions, the solution of Eq. (31*a*) for anisotropic dielectrical materials is *equivalent to* the equilibrium solution of anisotropic antiplane strain elasticity due to the parities between (electrical potential, electrical field, electrical induction, dielectrical tensor) and (displacement, stress, strain, elastic stiffness matrix). Additional partial different equations, for example, Eq. (30*a*), are required to solve the magnetic field.

Furthermore, the conclusions obtained in the previous subsections imply that all harmonic solutions, such as electrical potentials and antiplane strain displacements, belong to a subdomain of the family of plane stress solutions represented by biharmonic Airy's stress functions. This relationship seems also to be true for anisotropic cases. In Refs. [19,40] the general solution of anisotropic piezoelectric elasticity has been constructed based on Stroh's formulation [5], which demonstrates the same solution structure as the corresponding anisotropic elasticity problems obtained in Refs. [41,42].

In fact, according to the framework of classical elasticity theory [1,10,11], the constant mean in-plane stress rule is an additional constraint that requires the divergence of the Galerkin vector's

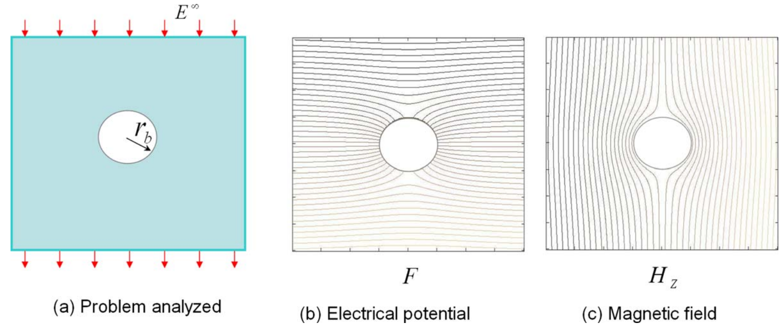

Fig. 2 Analytic solution of an infinitely large conductor plate containing a circular hole under a uniform electrical field E^∞ at remote: (*a*) model analyzed, (*b*) contours of the electrical potential, and (*c*) contours of the magnetic field

Laplacian to vanish, whereas the conjugate solution formula describes the procedure of the Papkovich–Neuber solution under this constraint. The recent published literature [43] shows that the Papkovich–Neuber general solution may also be applicable to anisotropic elasticity.

4 Examples

4.1 An Infinite Plate With Circular Hole Under a Uniform Electrical Load at Infinite. As an application example of the approach, the electrical-magnetic solution of a circular-shaped dielectric inclusion in an infinite conducting plate under uniform electrical load at a remote distance, depicted in Fig. 2(*a*), has been obtained using Eqs. (20), (21*a*), (21*b*), and (22)–(29). This is the case of an ellipsoid inclusion when its major and minor axes become equal [2].

The corresponding elastic solution has been described, e.g., in Ref. [6], where the Airy stress function for the configuration in Fig. 2(*a*) under a uniform remote stress, $\sigma_x=\sigma^\infty$, yields

$$U(z)=\frac{1}{2}[\bar{z}\phi(z)+z\bar{\phi}(z)] \quad \text{and} \quad \phi=\frac{\sigma^\infty}{4}\left(z+\frac{2r_b^2}{z}\right) \tag{32a}$$

By replacing σ^∞ with E^∞ in Eq. (30) and applying Eqs. (23) and (24), the "electrical Airy function" is obtained as follows:

$$U_E(z)=\frac{1}{2}[\bar{z}\phi^E(z)+z\bar{\phi}^E(z)], \quad \phi^E=\bar{\phi}, \quad \psi^E=0 \tag{32b}$$

Substituting Eq. (32*b*) into Eq. (23) and letting $z=x+iy$, $\bar{z}=x-iy$, then applying the congruity principle, the electrical potential, magnetic field, and density fields can be obtained as follows:

$$F=-\frac{E^\infty}{2}\left[x+\frac{x^3+y^2x+2xr_b^2}{x^2+y^2}\right] \tag{33a}$$

$$H_z=\frac{E^\infty}{2}\left[y+\frac{y^3+yx^2-2yr_b^2}{x^2+y^2}\right] \tag{33b}$$

$$E_x=\frac{E^\infty}{2}\left[1+\frac{3x^2+y^2+2r_b^2}{x^2+y^2}-\frac{2(x^3+y^2x+2xr_b^2)x}{(x^2+y^2)^2}\right] \tag{34}$$

$$E_y=E^\infty\left[\frac{xy}{x^2+y^2}-\frac{(x^3+y^2x+2xr_b^2)y}{(x^2+y^2)^2}\right] \tag{35}$$

where the $\{x,y\}$ coordinate system originates at the center of the circle, by which the x axis lies in the vertical direction of Fig. 2(*a*), defining the center line of symmetry of the infinite plate conductor. Figures 2(*b*) and 2(*c*) are the corresponding electrical potential and magnetic field. One can verify that

$$\nabla^2F=0, \quad \nabla^2H=0$$

4.2 Solutions for the Contact Problems in Figure 1(*b*). The problem to be analyzed in Fig. 1(*a*) represents two classes of physical contacts: (i) a semi-infinite conducting substrate is in contact with a conducting body under a static electrical load and (ii) a semi-infinite elastic substrate in contact with a rigid body under a mechanical load. The former can be classified into the boundary-value problem governed by Maxwell's equations for each individual conductor, for example, as illustrated by Fig. 1(*b*), the semi-infinite substrate with the boundary condition described by the contact surface. The latter has been investigated thoroughly, for example, in Refs. [5–7,12], with benchmark solutions for different boundary conditions. Applying the conjugate solution formula and the congruity principle introduced in Sec. 4.1, these benchmark solutions can be used to construct the solutions for the first class of problems with the boundary conditions on the contact surface, which can therefore be described by one of the following two expressions.

(b1) Given total electrical current flow I_Σ and electric potential on the contact surface $-a\le x\le a$, where the electrical potential can be expressed as a Taylor's expansion:

$$F(x)=a_0+a_1x+a_2x^2+\cdots \quad \text{for} \quad -a\le x\le a \tag{36}$$

(b2) Given a reference potential at any material point and electric field density $E_y(x)$ on the contact surface $-a\le x\le a$.

The condition (b1) corresponds to the displacement boundary condition in mechanical contact, whereas (b2) is the counterpart of the force boundary condition in linear elasticity. For (b1), the following additional boundary conditions are required.

$$I_\Sigma=\sigma^C\int_{-a}^{a}E_y dx=-\sigma^C\int_{-a}^{a}\frac{\partial F}{\partial y}dx \tag{37}$$

where I_Σ is the total electrical flow per unit thickness of the contact surface. Also, for both cases:

$$E_y(z)=0 \quad \text{when} \quad x>a \quad \text{or} \quad x<-a \tag{38a}$$

$$|E_x(z)+iE_y(z)|=0 \quad \text{when} \quad |z|\to\infty \tag{38b}$$

In order to demonstrate the procedure, the problem of Fig. 1(*b*) has been solved with boundary condition (b1) and the following symmetric condition:

$$F(x+iy)=F(-x+iy) \tag{39}$$

For simplification, $a=1$ and only the case of a constant electrical potential on contact surface, i.e., $F(x)=a_0$ is solved. Under this condition, the corresponding linear elastic contact solution has the stress function in the following form [6]:

$$\varphi(z) = -\frac{Pi}{2\pi}\ln\{z \pm \sqrt{z^2-1}\} \quad \text{and} \quad \varphi'(z) = \pm \frac{P}{2\pi\sqrt{1-z^2}} \tag{40}$$

where P is the total force perpendicular to the surface and the positive or negative sign in Eq. (40) corresponds to the contact with pressure or adhesion. Applying Eqs. (23) and (24) to formulate the electrical stress function, $\varphi_E(z)$, according to Eqs. (40) and (39),

$$\varphi_E = A[\log(\bar{z} + \sqrt{\bar{z}^2-1}) - \log(\bar{z} - \sqrt{\bar{z}^2-1})] \tag{41}$$

where A is a real constant; accordingly, the harmonic function, ψ_E, in the following form is chosen:

$$\psi_E = -2A[\sqrt{z^2-1}] \tag{42}$$

Substituting Eqs. (41) and (42) into Eq. (24) to obtain U_E and applying the following relations:

$$\frac{\partial U_E}{\partial x} = \frac{\partial U_E}{\partial z} + \frac{\partial U_E}{\partial \bar{z}}, \quad \frac{\partial U_E}{\partial y} = i\frac{\partial U_E}{\partial z} - i\frac{\partial U_E}{\partial \bar{z}} \tag{43}$$

Using the congruity principle,

$$-F + i\frac{H_z}{\sigma^C} = 2A\{\log(z + \sqrt{z^2-1}) - \log(z - \sqrt{z^2-1})\} + A_0 \tag{44}$$

where A_0 is a constant, and then applying Eqs. (3) and (43) to Eq. (44), the electrical current field yields

$$E_x - iE_y = \frac{4A}{\sqrt{z^2-1}} \tag{45}$$

Notice that on the contact surface, i.e., $y=0$, $|x| \le 1$:

$$z \pm \sqrt{z^2-1} = x \pm i\sqrt{1-x^2} \quad \text{and} \quad |x \pm i\sqrt{1-x^2}| \equiv 1 \tag{46a}$$

so

$$-F + i\frac{H_z}{\sigma^C} = 2Ai\left\{2\arctan\left(\frac{1-x^2}{x}\right) \pm n\pi\right\} + A_0, \quad n = 0,1,2,\ldots \tag{46b}$$

According to Eqs. (36) and (37), and symmetric condition (39), the constants A_0 and A are determined as follows:

$$A_0 = -a_0 - \frac{I_\Sigma}{2}, \quad A = \frac{I_\Sigma}{4\pi\sigma^C} \tag{47}$$

Applying the solution (44) to the case that electrical current flows from the substrate to the upper conductor, the corresponding normalized electrical potential and magnetic field are plotted in Figs. 3(*a*) and 3(*b*), where the $\tilde{H}_z$ varies from −1 to 1 inside the contact zone ($|x| \le 1$), whereas it remains constant along the remainder of the real axis. The electrical field (45) coincides with the stress distribution of the contact problem, the second relation of Eq. (40), solved in Ref. [6]. As illustrated by the Fig. 4(*a*), this function has two branches in its Riemann's surface, the half plane R_{I} corresponds to the Griffith's stress solution of center cracked infinite plate and the half plane R_{II} corresponds to the two-dimensional mechanical contact solution (40) and the electrical field defined by Eq. (44). The component E_y of the latter is plotted in Fig. 4(*b*).

Figure 5 is the contours of the electrical potential for of two semi-infinite conducts contact each other over the line segment $y=0$, $|x| \le 1$ according to Eq. (44), where the value of the conductivity σ^C in the upper semi-infinite conductor is a half of that in the lower plate. When the upper conductor is finite, for example, a rectangular bar, the corresponding stress function for the mechanical problem has been given, e.g., in Refs. [1,6,11]. It is straightforward to construct the corresponding electrical solution applying the procedure introduced.

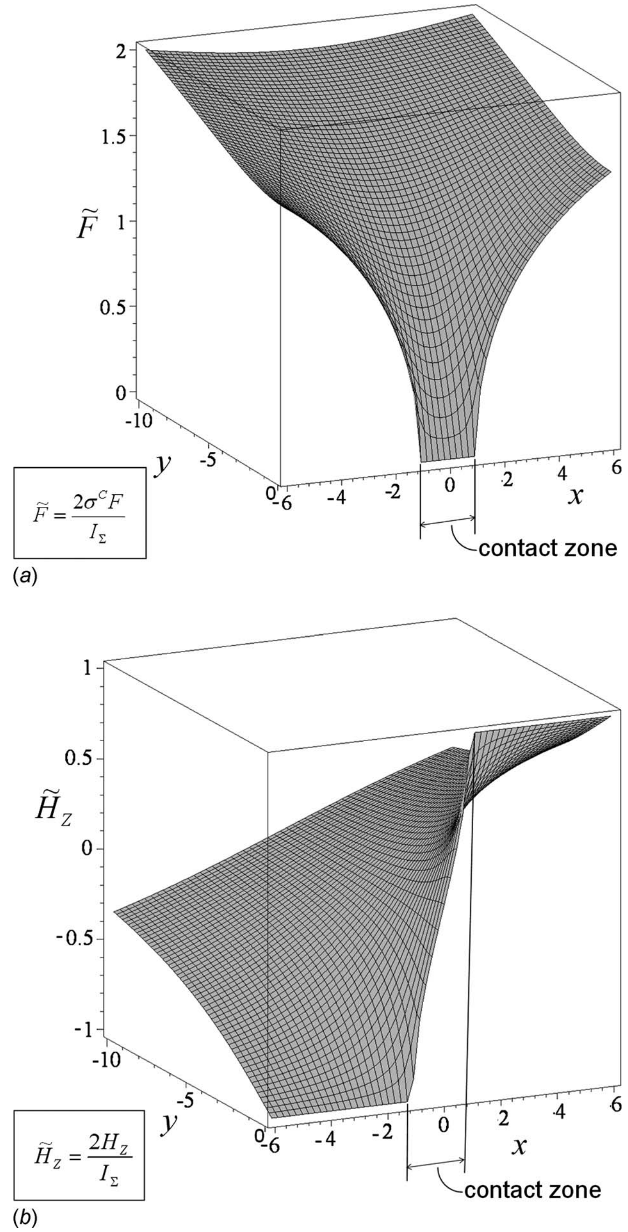

Fig. 3 The normalized electrical potential (*a*) and magnetic field (*b*) for the contact problem in Fig. 1(*b*)

5 Conclusions

This analysis reveals an intrinsic link between plane stress (or plane strain) linear elastic solution and two-dimensional electrostatic solution of Maxwell's equations. This linkage can be stated as follows:

1. Any plane stress/plane strain-displacement solution with constant mean in-plane stress can be represented by a pair of harmonic functions, so each displacement component is identical to a two-dimensional electrical potential with corresponding boundary condition. This is termed constant mean in-plane stress rule.
2. For any plane stress/plane strain solution, one can form a corresponding plane solution that satisfies the constant mean

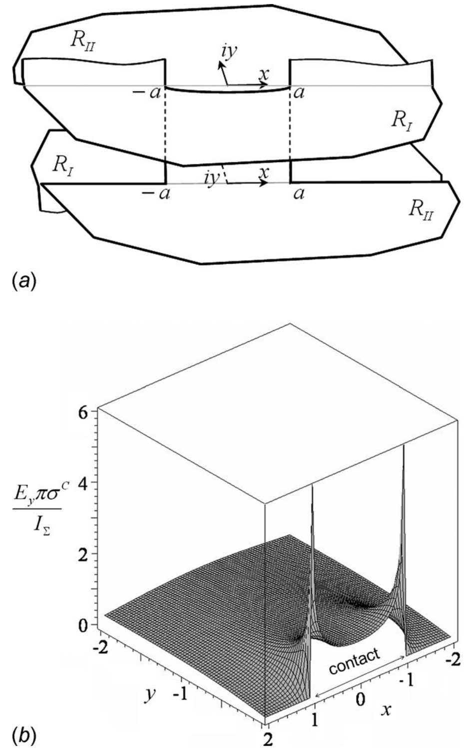

Fig. 4 Electrical field: (*a*) the two branches of the Riemann's surface of Eq. (45), where the branch R_{II} corresponds to the boundary value problem in Fig. 1(*b*), and (*b*) E_y according to Eq. (45)

Fig. 5 Contours of electrical potentials for the problem of contact between two semi-infinite conductors; on the contact surface $|x| \leq 1$, $y=0$ the electrical potential (36) is applied with $a_0=1, a_{i\neq 0}=0$, and the value of conductivity σ^C in upper semi-infinite conductor is one half of that in the lower plate

in-plane stress rule through the conjugate of its complex stress function. This procedure is termed conjugate solution formula.

3. The two displacement components of a plane stress/strain solution obtained according to the conjugate solution formula are structurally identical to a two-dimensional electrical field and accompanied magnetic field, which satisfy the electrostatic Maxwell's equations. This fact is termed congruity principle.

The congruity principle leads to a procedure to obtain analytical solutions of electrostatic boundary-value problem using Airy's stress function and Muskhelishvili's method. Examples of an infinite plate with a circular hole and a contact between a rigid conductor and a semi-infinite elastic conducting substrate have been analyzed.

According to the framework of the classic elasticity theory, the constant mean in-plane stress rule is an additional constraint that requires the divergence of Galerkin vector's Laplacian to vanish, whereas the conjugate solution formula describes the procedure of the Papkovich–Neuber solution under this constraint. The approach developed in this paper also leads to an alternate way to obtain close form solutions of antiplane strain elastic problems from plane stress/strain elastic solutions.

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Appendix: Proof of Equation (28)

When no out-plane current, $\sigma^C E_z = J_z = 0$, the Maxwell's equation IV in Eq. (1), $\nabla \times \boldsymbol{B} = \mu_H \boldsymbol{J}$, becomes [34]

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z_3} \\ -\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z_3} \\ \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \end{bmatrix} = \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \sigma^C \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \sigma^C \begin{bmatrix} -\frac{\partial F}{\partial x} \\ -\frac{\partial F}{\partial y} \\ 0 \end{bmatrix} \tag{A1}$$

where $\mathbf{H} = \mu_H \mathbf{B}$. Under the two-dimensional conditions:

$$E_z = H_x = H_y = 0 \tag{A2}$$

by substituting Eq. (A2) into Eq. (A1) one obtains

$$-\frac{\partial F}{\partial x} = \frac{\partial H_z}{\sigma^C \partial y}, \quad \frac{\partial F}{\partial y} = \frac{\partial H_z}{\sigma^C \partial x} \tag{A3}$$

which is the Cauchy–Riemann condition (14). Hence, F and corresponding H_z form a holomorphic function:

$$f(z) = -F + i\frac{H_z}{\sigma^C} \tag{28}$$

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